

Rate of Convergence of Major Cost Incurred in the In-Situ Permutation Algorithm

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ABSTRACT. The in-situ permutation algorithm due to MacLeod replaces (x_1, \dots, x_n) by $(x_{p(1)}, \dots, x_{p(n)})$ where $\pi = (p(1), \dots, p(n))$ is a permutation of $\{1, 2, \dots, n\}$ using at most $O(1)$ space. Kirshenhofer, Prodinger and Tichy have shown that the major cost incurred in the algorithm satisfies a recurrence similar to sequence of the number of key comparisons needed by the Quicksort algorithm to sort an array of n randomly permuted items. Further, Hwang has proved that the normalized cost converges in distribution. Here, following Neininger and Rüschendorf, we prove the that rate of convergence to be of the order $\Theta(\ln(n)/n)$ in the Zolotarev metric.

1. Introduction

The in-situ permutation algorithm developed by MacLeod [4] replaces (x_1, \dots, x_n) by $(x_{p(1)}, \dots, x_{p(n)})$ where $\pi = (p(1), \dots, p(n))$ is a permutation of $\{1, 2, \dots, n\}$ using at most $O(1)$ space. Kirshenhofer, Prodinger and Tichy [2] have shown that assuming the input comes from a sequence of independently and identically distributed random variables with a common continuous distribution, the major cost measures, say X_n , incurred in the algorithm, can be described by $X_0 = 0$, and for $n \geq 1$,

$$(1.1) \quad X_n \stackrel{d}{=} X_{I_n} + X_{n-1-I_n}^* + I_n,$$

where $(X_n), (X_n^*), (I_n)$ are independent, $X_n \stackrel{d}{=} X_n^*$, and I_n is uniformly distributed over $\{0, 1, \dots, n-1\}$. Here the symbol $\stackrel{d}{=}$ denotes equivalence in distribution.

The mean and variance of X_n were calculated by Knuth [3] which satisfy

$$\mathbf{E}(X_n) = n \ln n + (\gamma - 2)n + O(\ln n), \quad \text{Var}(X_n) = \sigma^2 n^2 - n \ln(n) + O(n)$$

where γ denotes Euler's constant and $\sigma := \sqrt{2 - \pi^2/6} > 0$.

Further, Hwang [1] showed using Rösler's contraction method that

$$Y_n := \frac{X_n - \mathbf{E}(X_n)}{n} \xrightarrow{d} Y$$

where \xrightarrow{d} denotes convergence in distribution. Here Y satisfies

$$(1.2) \quad Y \stackrel{d}{=} UY + (1 - U)Y^* + C(U)$$

where $Y \stackrel{d}{=} Y^*$, U is the uniform random variable over the unit interval, Y, Y^* , and U are independent, and $C(u) := (1 - u) \ln(1 - u) + u \ln(u) + u$.

We wish to estimate the rate of convergence $Y_n \rightarrow Y$ following Neininger and Rüschendorf [5]. The basic distance considered in [5] is the Zolotarev metric ζ_3 which given distributions $\mathcal{L}(V), \mathcal{L}(W)$ is defined by

$$\zeta_3(\mathcal{L}(V), \mathcal{L}(W)) := \sup_{f \in \mathcal{F}_3} |\mathbf{E}f(V) - \mathbf{E}f(W)|,$$

where $\mathcal{F}_3 := \{f \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}) : |f''(x) - f''(y)| \leq |x - y|\}$ is the space of all twice differentiable functions with second derivative being Lipschitz continuous with Lipschitz constant 1. Hereon we use the notation $\zeta_3(V, W) := \zeta_3(\mathcal{L}(V), \mathcal{L}(W))$. It is known that convergence in ζ_3 implies weak convergence and that $\zeta_3(V, W) < \infty$ if $\mathbf{E}V = \mathbf{E}W$, $\mathbf{E}V^2 = \mathbf{E}W^2$, and $\|V\|_3, \|W\|_3 < \infty$. The metric ζ_3 is ideal of order 3, that is, we have for T independent of (V, W) and $c \neq 0$

$$\zeta_3(V + T, W + T) \leq \zeta_3(V, W), \quad \zeta_3(cV, cW) = |c|^3 \zeta_3(V, W).$$

We wish to obtain following

THEOREM 1.1. *The major cost (X_n) incurred in the in-situ permutation algorithm satisfying recurrence (1.1) satisfies*

$$\zeta_3\left(\frac{X_n - \mathbf{E}(X_n)}{\sqrt{\text{Var}(X_n)}}, X\right) = \Theta\left(\frac{\log(n)}{n}\right), \quad (n \rightarrow \infty)$$

where $X := Y/\sigma$ is a scaled version of the limiting distribution in (1.2).

We modify the proof in [5] suited for the above case in the next section.

NOTATION: Subsequently, we use that $\text{Var}(Y) = \sigma$, $\|Y\|_3 < \infty$ where $\|Y\|_p := (\mathbf{E}|Y|^p)^{1/p}$, $1 \leq p < \infty$ denotes the L^p -norm.

2. The Proof

We start with the following lemma from [5].

LEMMA 2.1. *Let V, W have identical first and second moment with $\|V\|_3, \|W\|_3 < \infty$, then*

$$(2.1) \quad \frac{1}{6}|\mathbf{E}V^3 - \mathbf{E}W^3| \leq \zeta_3(V, W) \leq \frac{1}{6}(\|V\|_3^2 + \|V\|_3 \|W\|_3 + \|W\|_3^2)l_3(V, W)$$

where

$$(2.2) \quad l_p(\mathcal{L}(V), \mathcal{L}(W)) := l_p(V, W) := \inf\{\|V - W\|_p : V \stackrel{d}{=} V, W \stackrel{d}{=} W\}, \quad p \geq 1.$$

PROOF OF THEOREM 1.1. The constants $\sigma(n) \geq 0$ are defined by

$$(2.3) \quad \sigma^2(n) := \text{Var}(Y_n) = \sigma^2 - \frac{\ln(n)}{n} + O\left(\frac{1}{n}\right).$$

Lower Bound: Establishing the lower bounds only requires information of moments of (X_n) . Using the bound in lemma 2.1, we have

$$\zeta_3\left(\frac{X_n - \mathbf{E}(X_n)}{\sqrt{\text{Var}(X_n)}}, X\right) \geq \frac{1}{6} \left| \mathbf{E}\left(\frac{Y_n}{\sigma(n)}\right)^3 - \mathbf{E}\left(\frac{Y}{\sigma}\right)^3 \right|$$

Observe that the third moment of Y_n is

$$\mathbf{E}Y_n^3 = \frac{1}{n^3} \mathbf{E}(X_n - \mathbf{E}(X_n))^3 = \frac{1}{n^3} \kappa_3(X_n) = M_3 + O\left(\frac{1}{n}\right)$$

with $M_3 = \mathbf{E}(Y^3) > 0$ where we use the expansion of third cumulant $\kappa_3(X_n)$ of X_n which can be explicitly computed using generating functions for factorial moments with aid of Maple in [2]. The equation (2.3) gives us

$$\frac{1}{\sigma^3(n)} = \frac{1}{\sigma^3} + \frac{3}{2\sigma^5} \frac{\ln(n)}{n} + O\left(\frac{1}{n}\right),$$

and thus

$$\frac{1}{6} \left| \mathbf{E}\left(\frac{Y_n}{\sigma(n)}\right)^3 - \mathbf{E}\left(\frac{Y}{\sigma}\right)^3 \right| = \frac{M_3}{4\sigma^5} \frac{\ln(n)}{n} + O\left(\frac{1}{n}\right)$$

which proves the claimed lower bound in the theorem.

Upper Bound: The variates Y_n would satisfy the recurrence:

$$(2.4) \quad Y_n \stackrel{d}{=} \frac{I_n}{n} Y_{I_n} + \frac{n-1-I_n}{n} Y'_{n-1-I_n} + C_n(I_n), \quad n \geq 1,$$

where $(Y_n), (Y'_n), I_n$ are independent, $Y_k \stackrel{d}{=} Y'_k$ for all $k \geq 0$ and $C_n(k) := \frac{1}{n}(\mu(k) + \mu(n-1-k) - \mu(n) + k)$, with $\mu(n) := \mathbf{E}(X_n)$, $n \geq 0$. The rest of the proof follows identically as in the upper bound proof in [5]. \square

References

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